Irreducibility of the Punctual Quotient Scheme of a Surface

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Abstract

It is shown that the punctual quotient scheme Q_ℓ^r parametrizing all zero-dimensional quotients $\mathcal{O}_{\mathbb{A}^2}^{\oplus r} \to T$ of lenght ℓ and supported at some fixed point $0 \in \mathbb{A}^2$ in the plane is irreducible.

Let X be a smooth projective surface, E a locally free sheaf of rank $r \geq 1$ on X, and let $\ell \geq 1$ be an integer. $\mathrm{Quot}(E, \ell)$ denotes Grothendieck's quotient scheme [7] that parametrizes all quotients $E \to T$, where T is a zero-dimensional sheaf of length ℓ. Sending a quotient $E \to T$ to the point $\sum_{x \in X} \ell(T_x)x$ in the symmetric product $S^\ell(X)$ defines a morphism $\pi : \mathrm{Quot}(E, \ell) \to S^\ell(X)$ [7]. It is the purpose of this note to prove the following theorem:

Theorem 1 — $\mathrm{Quot}(E, \ell)$ *is an irreducible scheme of dimension* $\ell(r + 1)$. *The fibre of the morphism* $\pi : \mathrm{Quot}(E, \ell) \to S^\ell(X)$ *over a point* $\sum_x \ell_x x$ *is irreducible of dimension* $\sum_x (r\ell_x - 1)$.

Using the irreducibility result, one can check that a generic point in the fibre over $\ell x \in S^\ell(X)$ represents a quotient $E \to T$, where $T \cong \mathcal{O}_{X,x}/(s, t^\ell)$ and s and t are apropriately chosen local parameters in $\mathcal{O}_{X,x}$, i.e. T is the structure sheaf of a curvilinear subscheme in X.

If $r = 1$, i.e. if E is a line bundle, then $\mathrm{Quot}(E, \ell)$ is isomorphic to the Hilbert scheme $\mathrm{Hilb}^\ell(X)$. For this case, the first assertion of the theorem is due to Fogarty [5], whereas the second assertion was proved by Briançon [2]. For general $r \geq 2$, the first assertion of the theorem is a result due to J. Li and D. Gieseker [8],[6]. We give a different proof with a more geometric flavour, generalising a technique from Ellingsrud and Strømme [4]. The second assertion is a new result for $r \geq 2$. After finishing this paper we learned about a different approach by Baranovsky [1].

The natural generalizations of the theorem to higher dimensional or singular varieties are false, as is already apparent in the $r = 1$ case of the Hilbert schemes: The dimensions of the strata of quotients which are concentrated in some fixed point grow much faster with ℓ than the expected dimension of the 'generic' stratum.

1 Elementary Modifications

Let X be a smooth projective surface and $x \in X$. If N is a coherent $\mathcal{O}_X$-sheaf, $e(N_x) = \hom_X(N, k(x))$ denotes the dimension of the fibre $N(x)$, which by Nakayama's Lemma is the same as the minimal number of generators of the stalk N_x. If T is a coherent sheaf with zero-dimensional support, we denote by $i(T_x) = \hom_X(k(x), T)$ the dimension of the socle of T_x, i.e. the submodule $\mathrm{Soc}(T_x) \subset T_x$ of all elements that are annihilated by the maximal ideal in $\mathcal{O}_{X,x}$.

Lemma 2 — *Let $[q : E \to T] \in \mathrm{Quot}(E, \ell)$ be a closed point and let N be the kernel of q. Then the socle dimension of T and the number of generators of N at x are related as follows:*

$$e(N_x) = i(T_x) + r.$$

Proof. Write $e(N_x) = r + i$ for some integer $i \geq 0$. Then there is a minimal free resolution $0 \longrightarrow \mathcal{O}_{X,x}^i \xrightarrow{\alpha} \mathcal{O}_{X,x}^{r+i} \longrightarrow N_x \longrightarrow 0$, where all coefficients of the homomorphism α are contained in the maximal ideal of $\mathcal{O}_{X,x}$. We have $\mathrm{Hom}(k(x), T_x) \cong \mathrm{Ext}^1_X(k(x), N_x)$ and applying the functor $\mathrm{Hom}(k(x), \,.\,)$ one finds an exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_X(k(x), N_x) \longrightarrow \mathrm{Ext}^2_X(k(x), \mathcal{O}_{X,x}^i) \xrightarrow{\alpha'} \mathrm{Ext}^2_X(k(x), \mathcal{O}_{X,x}^{r+i}).$$

But as α has coefficients in the maximal ideal, the homomorphism α' is zero. Thus $\mathrm{Hom}(k(x), T) \cong \mathrm{Ext}^2_X(k(x), \mathcal{O}_{X,x}^i) \cong k(x)^i$. $\square$

The main technique for proving the theorem will be induction on the length of T. Let N be the kernel of a surjection $E \to T$, let $x \in X$ be a closed point, and let $\lambda : N \to k(x)$ be any surjection. Define a quotient $E \to T'$ by means of the following push-out diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & k(x) & \xrightarrow{\mu} & T' & \longrightarrow & T & \longrightarrow & 0 \\
 & & \lambda\uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & T & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & N' & = & N' & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

In this way every element $\langle\lambda\rangle \in \mathbb{P}(N(x))$ determines a quotient $E \to T'$ together with an element $\langle\mu\rangle \in \mathbb{P}(\mathrm{Soc}(T'_x)^\vee)$. (Here $W^\vee := \mathrm{Hom}_k(W, k)$ denotes the vector space dual to W.) Conversely, if $E \to T'$ is given, any such $\langle\mu\rangle$ determines $E \to T$ and a point $\langle\lambda\rangle$. We will refer to this situation by saying that T' is obtained from T by an elementary modification.

We need to compare the invariants for T and T': Obviously, $\ell(T') = \ell(T) + 1$. Applying the functor $\mathrm{Hom}(k(x), \,.\,)$ to the upper row in the diagram we get an exact sequence

$$0 \longrightarrow k(x) \longrightarrow \mathrm{Soc}(T'_x) \to \mathrm{Soc}(T_x) \longrightarrow \mathrm{Ext}^1_X(k(x), k(x)) \cong k(x)^2,$$

and therefore $|i(T_x) - i(T'_x)| \leq 1$. Moreover, we have $0 \leq e(T'_x) - e(T_x) \leq 1$. Two cases deserve closer inspection. Firstly, if e increases, then T' splits:

Lemma 3 — *Consider the natural homomorphisms $g : N(x) \to E(x)$ and $f : \mathrm{Soc}(T'_x) \to T'_x \to T'(x)$. The following assertions are equivalent*

1. $e(T'_x) = e(T_x) + 1$
2. $\langle\mu\rangle \notin \mathbb{P}(\ker(f)^\vee)$
3. $\langle\lambda\rangle \in \mathbb{P}(\mathrm{im}(g))$.

Moreover, if these conditions are satisfied, then $T' \cong T \oplus k(x)$ and $i(T'_x) = i(T_x) + 1$.

Proof. Clearly, $e(T'_x) = e(T_x) + 1$ if and only if $\mu(1)$ represents a non-trivial element in $T'(x)$ if and only if μ has a left inverse if and only if λ factors through E. $\square$

Secondly, if i increases for all modifications λ from T to any T', then the same phenomenon occurs for all 'backwards' modifications μ' from T to any T^-:

Lemma 4 — *Still keeping the notations above, let $E \to T'_\lambda$ be the modification of $E \to T$ determined by the point $\langle\lambda\rangle \in \mathbb{P}(N(x))$. Similarly, for $\langle\mu'\rangle \in \mathbb{P}(\mathrm{Soc}(T_x)^\vee)$ let $T^-_{\mu'} = T/\mu'(k(x))$. If $i(T'_{\lambda,x}) = i(T_x) + 1$ for all $\langle\lambda\rangle \in \mathbb{P}(N(x))$, then $i(T_x) = i(T^-_{\mu',x}) - 1$ for all $\langle\mu'\rangle \in \mathbb{P}(\mathrm{Soc}(T_x)^\vee)$ as well.*

Proof. Let $\Phi : \mathrm{Hom}_X(N, k(x)) \to \mathrm{Hom}_k(\mathrm{Ext}^1_X(k(x), N), \mathrm{Ext}^1_X(k(x), k(x)))$ be the homomorphism which is adjoint to the natural pairing

$$\mathrm{Hom}_X(N, k(x)) \otimes \mathrm{Ext}^1_X(k(x), N) \to \mathrm{Ext}^1_X(k(x), k(x)).$$

Identifying $\mathrm{Soc}(T_x) \cong \mathrm{Ext}^1_X(k(x), N)$, we see that $i(T'_{\lambda,x}) = 1 + i(T_x) - \mathrm{rank}(\Phi(\lambda))$. The action of $\Phi(\lambda)$ on a socle element $\mu' : k(x) \to T$ can be described by the following diagram of pull-backs and push-forwards

$$\begin{array}{ccccccccc}
0 & \to & N & \to & E & \to & T & \to & 0 \\
 & & \| & & \uparrow & & \uparrow \mu' & & \\
0 & \to & N & \to & N^-_{\mu'} & \to & k(x) & \to & 0 \\
 & & \lambda \downarrow & & \downarrow & & \| & & \\
0 & \to & k(x) & \to & \xi & \to & k(x) & \to & 0
\end{array}$$

The assumption that $i(T'_{\lambda,x}) = 1 + i(T_x)$ for all λ, is equivalent to $\Phi = 0$. This implies that for every μ' and every λ the extension in the third row splits, which in turn means that every λ factors through $N^-_{\mu'}$, i.e. that $N(x)$ embeds into $N^-_{\mu'}(x)$. Hence, for $T^-_{\mu'} = E/N^-_{\mu'} = \mathrm{coker}(\mu)$ we get $i(T^-_{\mu',x}) = e(N^-_{\mu',x}) - r = e(N_x) + 1 - r = i(T_x) + 1$. $\square$

2 The Global Case

Let $Y_\ell = \mathrm{Quot}(E, \ell) \times X$, and consider the universal exact sequence of sheaves on Y_ℓ:

$$0 \to \mathcal{N} \to \mathcal{O}_{\mathrm{Quot}} \otimes E \to \mathcal{T} \to 0.$$

The function $y = (s, x) \mapsto i(\mathcal{T}_{s,x})$ is upper semi-continuous. Let $Y_{\ell,i}$ denote the locally closed subset $\{y = (s, x) \in Y_\ell | i(\mathcal{T}_{s,x}) = i\}$ with the reduced subscheme structure.

Proposition 5 — *Y_ℓ is irreducible of dimension $(r+1)\ell + 2$. For each $i \geq 0$ one has $\mathrm{codim}(Y_{\ell,i}, Y_\ell) \geq 2i$,*

Clearly, the first assertion of the theorem follows from this.

Proof. The proposition will be proved by induction on ℓ, the case $\ell = 1$ being trivial: $Y_1 = \mathbb{P}(E) \times X$, the stratum $Y_{1,1}$ is the graph of the projection $\mathbb{P}(E) \to X$ and $Y_{1,i} = \emptyset$ for $i \geq 2$. Hence suppose the proposition has been proved for some $\ell \geq 1$.

We describe the 'global' version of the elementary modification discussed above. Let $Z = \mathbb{P}(\mathcal{N})$ be the projectivization of the family $\mathcal{N}$ and let $\varphi = (\varphi_1, \varphi_2) : Z \to$

$Y_\ell = \mathrm{Quot}(E,\ell) \times X$ denote the natural projection morphism. On $Z \times X$ there is canonical epimorphism

$$\Lambda : (\varphi_1 \times \mathrm{id}_X)^* \mathcal{N} \to (\mathrm{id}_Z, \varphi_2)_* \varphi^* \mathcal{N} \to (\mathrm{id}_Z, \varphi_2)_* \mathcal{O}_Z(1) =: \mathcal{K}.$$

As before we define a family $\mathcal{T}'$ of quotients of length $\ell+1$ by means of Λ:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{T}' & \longrightarrow & (\varphi_1, \mathrm{id}_X)^* \mathcal{T} & \longrightarrow & 0 \\
 & & \Lambda\uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & (\varphi_1, \mathrm{id}_X)^* \mathcal{N} & \longrightarrow & \mathcal{O}_Z \otimes E & \longrightarrow & (\varphi_1, \mathrm{id}_X)^* \mathcal{T} & \longrightarrow & 0
\end{array}$$

Let $\psi_1 : Z \to \mathrm{Quot}(E, \ell+1)$ be the classifying morphism for the family $\mathcal{T}'$, and define $\psi := (\psi_1, \psi_2 := \varphi_2) : Z \to Y_{\ell+1}$. The scheme Z together with the morphisms $\varphi : Z \to Y_\ell$ and $\psi : Z \to Y_{\ell+1}$ allows us to relate the strata $Y_{\ell,i}$ and $Y_{\ell+1,j}$. Note that $\psi(Z) = \bigcup_{j \geq 1} Y_{\ell+1,j}$.

The fibre of φ over a point $(s,x) \in Y_{\ell,i}$ is given by $\mathbb{P}(\mathcal{N}_s(x)) \cong \mathbb{P}^{r-1+i}$, since $\dim(\mathcal{N}_s(x)) = r + i(T_{s,x}) = r + i$ by Lemma 2. Similarly, the fibre of ψ over a point $(s',x) \in Y_{\ell+1,j}$ is given by $\mathbb{P}(\mathrm{Soc}(\mathcal{T}'_{s',x})^\vee) \cong \mathbb{P}^{j-1}$. If T' is obtained from T by an elementary modification, then $|i(T') - i(T)| \leq 1$ as shown above. This can be stated in terms of φ and ψ as follows: For each $j \geq 1$ one has:

$$\psi^{-1}(Y_{\ell+1,j}) \subset \bigcup_{|i-j| \leq 1} \varphi^{-1}(Y_{\ell,i}).$$

Using the induction hypothesis on the dimension of $Y_{\ell,i}$ and the computation of the fibre dimension of φ and ψ, we get

$$\dim(Y_{\ell+1,j}) + (j-1) \leq \max_{|i-j| \leq 1} \{(r+1)\ell + 2 - 2i + (r-1+i)\}$$

and

$$\dim(Y_{\ell+1,j}) \leq (r+1)(\ell+1) + 2 - 2j - \min_{|i-j| \leq 1} \{i - j + 1\}.$$

As $\min_{|i-j| \leq 1} \{i-j+1\} \geq 0$, this proves the dimension estimates of the proposition.

It suffices to show that Z is irreducible. Then $\mathrm{Quot}(E, \ell+1) = \psi_1(Z)$ and $Y_{\ell+1}$ are irreducible as well.

Since X is a smooth surface, the epimorphism $\mathcal{O}_{\mathrm{Quot}} \otimes E \to \mathcal{T}$ can be completed to a finite resolution

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{O}_{\mathrm{Quot}} \otimes E \longrightarrow \mathcal{T} \longrightarrow 0$$

with locally free sheaves $\mathcal{A}$ and $\mathcal{B}$ on Y_ℓ of rank n and $n+r$, respectively, for some positive integer n. It follows that $Z = \mathbb{P}(\mathcal{N}) \subset \mathbb{P}(\mathcal{B})$ is the vanishing locus of the composite homomorphism $\varphi^* \mathcal{A} \to \varphi^* B \to \mathcal{O}_{\mathbb{P}(\mathcal{B})}(1)$. In particular, assuming by induction that Y_ℓ is irreducible, Z is locally cut out from an irreducible variety of dimension $(r+1)\ell + 2 + (r+n-1)$ by n equations. Hence every irreducible component of Z has dimension at least $(r+1)(\ell+1)$. But the dimension estimates for the stratum $Y_{\ell,i}$ and the fibres of φ over it yield:

$$\dim(\varphi^{-1}(Y_{\ell,i})) \leq (r+1)\ell + 2 - 2i + (r+i-1) = (r+1)(\ell+1) - i,$$

which is strictly less than the dimension of any possible component of Z, if $i \geq 1$. This implies that the irreducible variety $\varphi^{-1}(Y_{\ell,0})$ is dense in Z. Moreover, since the fibre of ψ over $Y_{\ell+1,1}$ is zero-dimensional, $\dim(Y_{\ell+1}) = \dim(Y_{\ell+1,1}) + 2 = \dim(Z) + 2$ has the predicted value. $\square$

3 The Local Case

We now concentrate on quotients $E \to T$, where T has support in a single fixed closed point $x \in X$. For those quotients the structure of E is of no importance, and we may assume that $E \cong \mathcal{O}_X^r$. Let Q_ℓ^r denote the closed subset

$$\left\{[\mathcal{O}_X^r \to T] \in \mathrm{Quot}(\mathcal{O}_X^r, \ell) |\ \mathrm{Supp}(T) = \{x\}\right\}$$

with the reduced subscheme structure. We may consider Q_ℓ^r as a subscheme of $Y_{\ell,1}$ by sending $[q]$ to $([q], x)$. Then it is easy to see that $\varphi^{-1}(Q_\ell^r) = \psi^{-1}(Q_{\ell+1}^r)$. Let this scheme be denoted by Z'.

We will use a stratification of Q_ℓ^r both by the socle dimension i and the number of generators e of T and denote the corresponding locally closed subset by $Q_{\ell,i}^{r,e}$. Moreover, let $Q_{\ell,i}^r = \bigcup_e Q_{\ell,i}^{r,e}$ and define $Q_\ell^{r,e}$ similarly. Of course, $Q_{\ell,i}^{r,e}$ is empty unless $1 \leq i \leq \ell$ and $1 \leq e \leq \min\{r, \ell\}$.

To prove the second half of the theorem it suffices to show:

Proposition 6 — *Q_ℓ^r is an irreducible variety of dimension $r\ell - 1$.*

Lemma 7 — $\dim(Q_{\ell,i}^{r,e}) \leq (r\ell - 1) - (2(i-1) + \binom{e}{2})$.

Proof. By induction on ℓ: if $\ell = 1$, then $Q_1^r \cong \mathbb{P}^{r-1}$, and $Q_{1,i}^{r,e} = \emptyset$ if $e \geq 2$ or $i \geq 2$. Assume that the lemma has been proved for some $\ell \geq 1$.

Let $[q' : \mathcal{O}_X^r \to T'] \in Q_{\ell+1,j}^{r,e}$ be a closed point. Suppose that the map $\mu : k(x) \to T'(x)$ represents a point in $\psi^{-1}([q']) = \mathbb{P}(\mathrm{Soc}(T'_x)^\vee)$ and that $T_\mu = \mathrm{coker}(\mu)$ is the corresponding modification. If $i = i(T_{\mu,x})$ and $\varepsilon = e(T_{\mu,x})$, then, according to Section 1, the pair (i, ε) can take the following values:

$$(i, \varepsilon) = (j-1, e-1),\ (j-1, e),\ (j, e)\ \text{ or }\ (j+1, e), \tag{1}$$

in other words:

$$\psi^{-1}(Q_{\ell+1,j}^{r,e}) \subset \varphi^{-1}(Q_{\ell,j-1}^{r,e-1}) \cup \bigcup_{|i-j|\leq 1} \varphi^{-1}(Q_{\ell,i}^{r,e}).$$

Subdivide $A = Q_{\ell,j}^{r,e}$ into four locally closed subsets $A_{i,\varepsilon}$ according to the generic value of (i, ε) on the fibres of ψ. Then

$$\dim(A_{i,\varepsilon}) + (j-1) \leq \dim(Q_{\ell,i}^{r,\varepsilon}) + d_{i,\varepsilon},$$

where $d_{i,\varepsilon}$ is the fibre dimension of the morphism

$$\varphi : \psi^{-1}(A_{i,\varepsilon}) \cap \varphi^{-1}(Q_{\ell,i}^{r,\varepsilon}) \longrightarrow Q_{\ell,i}^{r,\varepsilon}.$$

By the induction hypothesis we have bounds for $\dim(Q_{\ell,i}^{r,\varepsilon})$, and we can bound $d_{i,\varepsilon}$ in the four cases (1) as follows:

A) Let $[q : \mathcal{O}_X^r \to T] \in Q_{\ell,j-1}^{r,e-1}$ be a closed point with $N = \ker(q)$. As we are looking for modifications T' with $e(T'_x) = e$, we are in the situation of Lemma 3 and may conclude

$$\begin{aligned} \varphi^{-1}([q]) \cap \psi^{-1}(A_{e-1,j-1}) &\cong \mathbb{P}(\mathrm{im}(g : N(x) \to k(x)^r)) \\ &\cong \mathbb{P}(\ker(k(x)^r \to T(x))) \cong \mathbb{P}^{r-e}, \end{aligned}$$

since $\mathrm{im}(k(x)^r \to T(x)) \cong k^{e-1}$. Hence $d_{j-1,e-1} = r - e$ and

$$\begin{aligned}\dim(A_{j-1,e-1}) &\leq \dim Q^{r,e-1}_{\ell,j-1} + (r-e) - (j-1) \\ &\leq \left\{(r\ell - 1) - 2(j-2) - \binom{e-1}{2}\right\} + (r-e) - (j-1) \\ &= \left\{(r(\ell+1) - 1) - 2(j-1) - \binom{e}{2}\right\} - (j-2).\end{aligned}$$

Note that this case only occurs for $j \geq 2$, so that $(j-2)$ is always nonnegative.

B) In the three remaining cases

$$\varepsilon = e \text{ and } i = j-1,\ j,\ \text{ or } j+1$$

we begin with the rough estimate $d_{i,e} \leq r + i - 1$ as in Section 2. This yields:

$$\begin{aligned}\dim(A_{i,e}) &\leq \left\{(r\ell - 1) - 2(i-1) - \binom{e}{2}\right\} + (r+i-1) - (j-1) &(2)\\ &= \left\{(r(\ell+1) - 1) - 2(j-1) - \binom{e}{2}\right\} - (i-j). &(3)\end{aligned}$$

Thus, if $i = j$ we get exactly the estimate asserted in the Lemma, if $i = j+1$ the estimate is better than what we need by 1, but if $i = j-1$, the estimate is not good enough and fails by 1. It is this latter case that we must further study: let $[q : \mathcal{O}_X^r \to T]$ be a point in $Q^{r,e}_{\ell,j-1}$ with $N = \ker(q)$. There are two alternatives:

— *Either* the fibre $\varphi^{-1}([q]) \cap \psi^{-1}(A_{j-1,e})$ is a *proper* closed subset of $\mathbb{P}(N(x))$ which improves the estimate for the dimension of the fibre $\varphi^{-1}([q])$ by 1,

— *or* this fibre *equals* with $\mathbb{P}(N(x))$, which means that the socle dimension increases for all modifications of T. In this case we conclude from Lemma 4 that also $i(T^-) = i(T)+1$ for every modification $T^- = \mathrm{coker}(\mu^- : k(x) \to T)$. But, as we just saw, calculation (3), applied to the contribution of $Q^{r,e}_{\ell-1,j}$ to $Q^{r,e}_{\ell,j-1}$, shows that the dimension estimate for the locus of such points $[q]$ in $Q^{r,e}_{\ell,j-1}$ can be improved by 1 compared to the dimension estimate for $Q^{r,e}_{\ell,j-1}$ as stated in the lemma.

Hence in either case we can improve estimate (3) by 1 and get

$$\dim(A_{j-1,e}) \leq (r(\ell+1) - 1) - 2(j-1) - \binom{e}{2}$$

as required. Thus, the lemma holds for $\ell + 1$. □

Lemma 8 — $\psi(\varphi^{-1}(Q^{r,e}_\ell)) \subset \overline{Q^{r,e}_{\ell+1}}$.

Proof. Let $[q : \mathcal{O}_X^r \to T] \in Q^{r,e}_{\ell,i}$ be a closed point with $N = \ker(q)$. Then $\varphi^{-1}([q]) = \mathbb{P}(N(x)) \cong \mathbb{P}^{r+i-1}$ and $\varphi^{-1}([q]) \cap \psi^{-1}(Q^{r,e+1}_{\ell+1}) \cong \mathbb{P}(\mathrm{im}(G)) \cong \mathbb{P}^{r-e-1}$. Since we always have $e \geq 1, i \geq 1$, a dense open part of $\varphi^{-1}([q])$ is mapped to $Q^{r,e}_{\ell+1}$. □

Lemma 9 — *If $r \geq 2$ and if Q^{r-1}_ℓ is irreducible of dimension $(r-1)\ell - 1$, then $Q^{r,<r}_\ell := \bigcup_{e<r} Q^{r,e}_\ell$ is an irreducible open subset of Q^r_ℓ of dimension $r\ell - 1$.*

Proof. Let M be the variety of all $r \times (r-1)$ matrices over k of maximal rank, and let $0 \to \mathcal{O}_M^{r-1} \to \mathcal{O}_M^r \to \mathcal{L} \to 0$ be the corresponding tautological sequence of locally free sheaves on M. Consider the open subset $U \subset M \times Q_\ell^r$ of points $(A, [\mathcal{O}^r \to T])$ such that the composite homomorphism

$$\mathcal{O}^{r-1} \xrightarrow{A} \mathcal{O}^r \longrightarrow T$$

is surjective. Clearly, the image of U under the projection to Q_ℓ^r is $Q_\ell^{r,<r}$. On the other hand, the tautological epimorphism

$$\mathcal{O}_{U\times X}^{r-1} \to \mathcal{O}_{U\times X}^r \to (\mathcal{O}_M \otimes \mathcal{T})|_{U\times X}$$

induces a classifying morphism $g' : U \to Q_\ell^{r-1}$. The morphism $g = (pr_1, g') : U \to M \times Q_\ell^{r-1}$ is surjective. In fact, it is an affine fibre bundle with fibre

$$g^{-1}(g(A, [\mathcal{O}^{r-1} \to T])) \cong \mathrm{Hom}_k(\mathcal{L}(A), T) \cong \mathbb{A}_k^\ell.$$

Since Q_ℓ^{r-1} is irreducible of dimension $(r-1)\ell - 1$ by assumption, U is irreducible of dimension $r\ell - 1 + \dim(M)$, and $Q_\ell^{r,<r}$ is irreducible of dimension $r\ell - 1$. $\square$

Proof of Proposition 6. The irreducibility of Q_ℓ^r will be proved by induction over r and ℓ: the case ($\ell = 1, r$ arbitrary) is trivial; whereas (ℓ arbitrary $, r = 1$) is the case of the Hilbert scheme, for which there exist several proofs ([2], [4]). Assume therefore that $r \geq 2$ and that the proposition holds for (ℓ, r) and $(\ell+1, r-1)$. We will show that it holds for $(\ell+1, r)$ as well.

Recall that $Z' := \varphi^{-1}(Q_\ell^r) = Q_\ell^r \times_{Y_\ell} Z$. Every irreducible component of Z' has dimension greater than or equal to $\dim(Q_\ell^r) + r - 1 = r(\ell+1) - 2$ (cf. Section 2). On the other hand, $\dim(\varphi^{-1}(Q_{\ell,i}^r)) \leq r\ell - 1 - 2(i-1) + (r+i-1) = r(\ell+1) - i$. Thus an irreducible components of Z' is either the closure of $\varphi^{-1}(Q_{\ell,1}^r)$ (of dimension $r(\ell+1) - 1$)) or the closure of $\varphi^{-1}(W)$ for an irreducible component $W \subset Q_{\ell,2}^r$ of maximal possible dimension $r\ell - 3$. But according to Lemma 8 the image of $\varphi^{-1}(W)$ under ψ will be contained in the closure of $Q_{\ell+1}^{r,<r}$, unless W is contained in $Q_{\ell,2}^{r,r}$. But Lemma 7 says that $Q_{\ell,2}^{r,r}$ has codimension $\geq 2 + \binom{r}{2} \geq 3$ if $r \geq 2$, and hence cannot contain W for dimension reasons. Hence any irreducible component of Z' is mapped by ψ into the closure of $Q_{\ell+1}^{r,<r}$ which is irreducible by Lemma 9 and the induction hypothesis. This finishes the proof of the proposition. $\square$

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